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DIFFERENT APPROACHES IN THE CONSTRUCTIVE MARTINGALE REPRESENTATION OF BROWNIAN FUNCTIONALS

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РІЗНІ ПІДХОДИ В КОНСТРУКТИВНОМУ МАРТИНГАЛЬНОМУ ПРЕДСТАВЛЕННІ БРОУНІВСЬКИХ ФУНКЦІОНАЛІВ

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Abstract. In this work, we study the issues of a constructive stochastic integral representation of Brownian functionals, which are interesting from the point of view of their practical application in the problem of hedging a European option. In addition to briefly discussing known results in this direction, in the case of stochastically smooth (in Malliavin sense) functionals, we also illustrate the usefulness of the Glonti–Purtukhia representation for non-smooth functionals. In particular, we generalize the Clarke–Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable, and find an explicit expression for the integrand. Moreover, we consider such functionals that do not satisfy even weakened conditions, that is, non-smooth, past-dependent Brownian functionals, the conditional mathematical expectations of which are also not stochastically differentiable, and again we give a constructive martingale representation.

Keywords: martingale representation, Malliavin derivative, Clark–Ocone formula, Glonti–Purtukhia representation.

Анотація. У цій роботі досліджуються питання конструктивного стохастичного інтегрального зображення броунівських функціоналів, цікаві з погляду їх практичного застосування в задачі

хеджування європейського опціону. Крім короткого обговорення відомих результатів у цьому напрямку, у разі стохастично гладких (у сенсі Маллявена) функціоналів, ми також проілюструємо корисність зображення Глонті–Пуртухії для негладких функціоналів. Зокрема, ми узагальнюємо формулу Кларка–Оконе на випадок, коли функціонал не є стохастично гладким, але його умовне математичне сподівання стохастично диференційовне і знаходимо явний вираз для підінтегральної функції. Більш того, ми розглядаємо такі функціонали, які не задовольняють навіть ослабленим умовам, тобто негладкі, броунівські функціонали, що залежать від траєкторії, умовні математичні сподівання яких також не є стохастично диференційовними, і знову даємо конструктивне мартингальне зображення.
Ключові слова: мартингальне зображення, похідна Маллявена, формула Кларка–Окона, зображення Глонті–Пуртухії.

1. Introduction

Let B_t be a Brownian motion on a standard filtered probability space

$$(\Omega, \Im, \Im_t, P)$$

and let $\Im_t = \Im_t^B$ be the augmentation of the filtration generated by B.

Definition 1. Let H be the class of functions $f : [0, T] \times \Omega \longrightarrow R$ such that
(i) the mapping $(t, \omega) \longrightarrow f(t, \omega)$ is $\mathcal{B}([0, T]) \otimes \Im$-measurable;
(ii) for all $t \in [0, T]$ the random variable $f(t, \cdot)$ is $\Im_t$-measurable;
(iii) $\int_\Omega [\int_0^T f^2(t, \omega) dt] dP(\omega) < \infty$.

The stochastic integral as a process of a function of the space H has an important property (see, for example, [1]):

Proposition 1. *If $f \in H$ then the stochastic process*

$$\xi_t = \int_0^t f(s, \omega) dB_s(\omega)$$

is a martingale with respect to the filtration $\{\Im_t\}$.

On the other hand, according to the well-known Clark formula [2], the inverse statement (so-called martingale representation theorem) is also true[1]:

Theorem 1 ([2])**.** *If F is a square integrable $\Im_T$-measurable random variable, then (due to the Clark formula) there exist a square integrable $\Im_t$-adapted random process $\varphi(t, \omega)$ such that*

$$F = EF + \int_0^T \varphi(t, \omega) dB_t(\omega).$$

[1]It should be noted that the first proof of the martingale representation theorem was implicitly provided by Ito himself (see, [3]).

Taking the conditional mathematical expectation from the both sides of the last relation we obtain that for the associated to F Levy's martingale

$$M_t = E[F|\Im_t]$$

the following stochastic integral representation is true

$$M_t = M_0 + \int_0^t \varphi(s,\omega)dB_s(\omega).$$

It should be noted that this problem is closely related to important questions in applications such as finding hedge portfolios in finance[2]. Therefore, it is very important to find an explicit expression of the integrand for the stochastic integral, but Clark's theorem does not provide such a possibility. In many subsequent works using the Malliavin calculus or some kind of differential calculus for random processes, the results are also quite general, but unsatisfactory from the point of view of explicitness: the integrands in stochastic integral representations always include predictable projections or conditional mathematical expectations and some kinds of gradients.

2. Preliminaries

Finding an explicit expression for $\varphi(t,\omega)$ is a very difficult task. There is one general result in this direction, called the Clark–Ocone formula [5], according to which $\varphi(t,\omega) = E[D_tF|\Im_t]$, where D_t is the so-called Malliavin stochastic derivative.

Definition 2. The class of smooth Brownian functionals S is the class of a random variables which has the form

$$F = f(B_{t_1},...,B_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0,T],\ n \geq 1,$$

where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

Definition 3 ([6])**.** The stochastic (Malliavin) derivative of a smooth random variable $F \in S$ is the stochastic process D_tF given by

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1},...,B_{t_n})I_{[0,t_i]}(t).$$

In fact, we have defined the Malliavin derivative as an «inverse» of the Ito stochastic integral (with deterministic integrand) in the sense that $DB(h) = h$ (where

$$B(h) := \int_0^T h(s)dB_s \ \text{ and } \ D_t\Big[\int_0^T h(s)dB_s\Big] = h(t),$$

[2]In the 80th of the past century, it turned out (see, [4]) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different type.

as well as it's clear that $B_\theta = B(I_{[0,\theta]}(\cdot))$ and $D_t B_\theta = I_{[0,\theta]}(t))$.

Definition 4 (see, [6])**.** Operator $D.$ is closable as an operator from the space $L_2(\Omega)$ to the space $L_2(\Omega; L_2([0,T]))$. We will denote its domain by $D_{2,1}$. That means, $D_{2,1}$ is equal to the adherence of the class of smooth random variables with respect to the norm

$$||F||_{2,1} = \{E[|F|^2 + (||D.F||^2_{L_2([0,T])})]\}^{1/2}.$$

Theorem 2 (see, [5])**.** *If F is differentiable in Malliavin sense, $F \in D_{2,1}$, then the following stochastic integral representation is fulfilled*

$$F = E[F] + \int_0^T E[D_t F|\Im_t]\mathrm{d}B_t.$$

A different method for finding the process $\varphi(t,\omega)$ was proposed by Shiryaev and Yor [7] and Graversen, Shiryaev and Yor [8], which was based on the Ito (generalized) formula and the Levy theorem for the Levy martingale $M_t = E[F|\Im_t]$ associated with F. In particular, they proposed a method that gives an explicit martingale representation of the so-called running maximum of Brownian motion. Later on, using the Clark–Ocone formula, Renaud and Remillard [9] established an explicit martingale representation for path-dependent Brownian functionals, in particular, they considered a continuously differentiable function of three stochastically smooth quantities: from Brownian motion with drift and processes of its maximum and minimum (a direct consequence of the obtained representation is an explicit martingale representation of the geometric Brownian motion).

Despite the fact that the Clarke–Ocone formula gives the construction of the integrand, there are problems with practical implementation. In particular, even in the case of smoothness F, the calculation of its Malliavin derivative, and then the conditional mathematical expectation (or predictable projection in the general case) of the obtained expression is very difficult.

Note now that in all the cases mentioned above, the functionals under study are stochastically (in Malliavin sense) smooth. On the other hand, the second author of this article, together with Dr. V. Jaoshvili (see, [10]), in the framework of the classical Ito's calculus, allows us to construct $\varphi(t,\omega)$ explicitly, using both the standard L_2 theory and the theory of weighted Sobolev spaces, for some class of Brownian functionals that do not have a stochastic derivative.

Further, the past-dependent nons-mooth Brownian functionals of a special type were considered in the works of Glonti and Purtukhia ([11, 12, 13, 14]), Glonti, Jaoshvili and Purtukhia [15] and Livinska and Purtukhia [16], where they developed methods for obtaining an integral representation using the Trotter–Meyer theorem[3] or directly calculating the probability distributions of the corresponding functionals. In particular, at the first stage, the Clark stochastic integral representation for local time is derived, and then, based on

[3]The Trotter–Meyer theorem (see, [17]) establishes a relationship between the predictable square variation of a semimartingale and its local time.

the Trotter–Meyer theorem and the Fubini stochastic type theorem, the Clark integral representation is obtained with an explicit form of the integrand.

Later it turned out that the requirement of smoothness of the functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. The second author of this article, together with prof. O. Glonti in [18] considered Brownian functionals that are not stochastically differentiable and generalized the Clark–Ocone formula.

Theorem 3 (see, [18])**.** *Suppose that $G_t = E(F|\Im_t)$ is Malliavin differentiable ($G_t(\cdot) \in D_{2,1}$) for almost all $t \in [0,T)$. Then we have the stochastic integral representation*

$$G_T = F = EF + \int_0^T \nu_s \mathrm{d}B_s \quad (P - a.s.),$$

where

$$\nu_s := \lim_{t\uparrow T} E[D_s G_t|\Im_s] \quad in \ the \quad L_2([0,T]\times\Omega).$$

Remark 1. To calculate the conditional mathematical expectation, we need the transition probability of Brownian motion

$$P\{B_t \in A|\Im_s\} = p(s,t,B_s,A),$$

where $0 \le s \le t$, A is the Borel subset of the space R^1, and

$$p(s,t,x,A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A \exp\{-\frac{(x-y)^2}{2(t-s)} dy\}.$$

It is well known that for all measurable bounded functions f we have

$$E[f(B_t)|\Im_s] = \int_{R^1} f(y)p(s,t,B_s,dy).$$

Remark 2. It should be noted, that if random variable is stochastically differentiable in Malliavin sense, then its conditional mathematical expectation is differentiable too (see, Proposition 1.2.8 [19]). On the other hand, it is possible that conditional mathematical expectation can be smooth even if random variable is not stochastically smooth. For example, it is well-known that $I_{\{B_T\le 0\}} \notin D_{2,1}$[4], but for all $t \in [0,T)$, it is not difficult to see that (see Remark 1):

$$E[I_{\{B_T\le 0\}}|\Im_t] = \Phi\left(\frac{-B_t}{\sqrt{T-t}}\right) \in D_{2,1},$$

where Φ is standard normal distribution function.

Here we illustrate the usefulness of the Glonti–Purtukhia representation for non-smooth Brownian functionals. In particular, we give a constructive martingale representation when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable. Moreover, we consider such functionals that do not satisfy even the weakened Glonti–Purtukhia conditions (see, Theorem 3), that is, a non-smooth, past-dependent

[4]It should be noted that indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one (see, Proposition 1.2.6 [19]).

Brownian functional, the conditional mathematical expectation of which is also not stochastically differentiable, and we find the integrand of its martingale representation.

3. Main Results

Theorem 4. *For any non-negative constant $C \geq 0$ the non-smooth Brownian functional $F(C) = I_{\{B_T^+ \leq C\}}$*[5] *admits the following stochastic integral representation*

$$I_{\{B_T \leq C\}} = \Phi\left(\frac{C}{\sqrt{T}}\right) - \int_0^T \frac{1}{\sqrt{T-s}}\varphi\left(\frac{C-B_s}{\sqrt{T-s}}\right) \mathrm{d}B_s,$$

where φ is standard normal distribution density function.

Proof. It is clear that for any $C \geq 0$

$$EF(C) = P\{B_T^+ \leq C\} = P\{B_T \leq C\} = \Phi\left(\frac{C}{\sqrt{T}}\right).$$

Further, for all $t \in [0,T)$, due to the Remark 1, we have

$$G_t := G_t(C) = E[F(x)|\Im_t] = \int_{-\infty}^{\infty} I_{\{y^+ \leq C\}} p(t,T,B_t,dy) =$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} I_{\{y \leq C\}} \exp\left\{-\frac{(y-B_t)^2}{2(T-t)}\right\} dy = \Phi\left(\frac{C-B_t}{\sqrt{T-t}}\right).$$

Therefore, it is clear that $G_t \in D_{2,1}$. Hence, according to the rule of stochastic differentiation of a composite function (see, Proposition 1.2.3 [19]) the Malliavin derivative of the functional G_t ($t \in [0,T)$) has the form

$$D_s G_t = -I_{[0,t]}(s) \frac{1}{\sqrt{T-t}} \varphi\left(\frac{C-B_t}{\sqrt{T-t}}\right).$$

Therefore, again, thanks to Remark 1, using the standard integration technique (including the identity:

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} dz = \sqrt{2\pi}\sigma,$$

[5] $B_T^+ := max\{B_T, 0\}$.

it is not difficult to see that

$$\begin{aligned}
E(D_sG_t|\Im_s) &= -I_{[0,t]}(s)\frac{1}{\sqrt{T-t}}E[\varphi(\frac{C-B_t}{\sqrt{T-t}})|\Im_s] = \\
&= -\frac{I_{[0,t]}(s)}{\sqrt{T-t}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}\varphi(\frac{C-y}{\sqrt{T-t}})\exp\{-\frac{(y-B_s)^2}{2(t-s)}\}\mathrm{d}y = \\
&= -I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(t-s)}}\frac{1}{\sqrt{2\pi(T-t)}}\times \\
&\times \int_{-\infty}^{\infty}\exp\{-\frac{(C-y)^2}{2(T-t)}\}\exp\{-\frac{(y-B_s)^2}{2(t-s)}\}\mathrm{d}y = \\
&= -I_{[0,t]}(s)\exp\{-\frac{(C-B_s)^2}{2(T-s)}\}\frac{1}{\sqrt{2\pi(t-s)}}\frac{1}{\sqrt{2\pi(T-t)}}\times \\
&\times \int_{-\infty}^{\infty}\exp\{-\frac{T-s}{2(T-t)(t-s)}[y-\frac{C(t-s)+B_s(T-t)}{T-s}]^2\}\mathrm{d}y = \\
&= -I_{[0,t]}(s)\exp\{-\frac{(C-B_s)^2}{2(T-s)}\}\frac{1}{\sqrt{2\pi(t-s)}}\frac{1}{\sqrt{2\pi(T-t)}}\times \\
&\times \sqrt{2\pi}\sqrt{\frac{(T-t)(t-s)}{T-s}} = -I_{[0,t]}(s)\frac{1}{\sqrt{T-s}}\varphi(\frac{C-B_s}{\sqrt{T-s}}).
\end{aligned}$$

Further, it is evident that in this case there exists a sequence $t_n \in [0,T)$, $t_n \uparrow T$, such that

$$\nu_s := \lim_{t\uparrow T} E[D_sG_t|\Im_s] =$$

$$= -I_{[0,T]}(s)\frac{1}{\sqrt{T-s}}\varphi(\frac{C-B_s}{\sqrt{T-s}}) \quad in \quad L_2([0,T]\times\Omega),$$

which, based on Theorem 3, completes the proof of the theorem. □

Corollary 1. *For any positive $C_1 > 0$ and non-negative constants $C_2 \geq 0$ and $C \geq 0$ the non-smooth Brownian functional $F(C_1,C_2,C) = I_{\{(C_1B_T+C_2)^+\leq C\}}$ allows the following stochastic integral representation*

$$I_{\{(C_1B_T+C_2)^+\leq C\}} = \Phi\left(\frac{C-C_2}{C_1\sqrt{T}}\right) - \int_0^T \frac{1}{\sqrt{T-s}}\varphi\left(\frac{C-C_2-C_1B_s}{C_1\sqrt{T-s}}\right)\mathrm{d}B_s.$$

Consider now the Brownian functional of integral type

$$F = \int_0^T I_{\{B_t^+\leq C\}}\mathrm{d}t.$$

Remark 3. Despite the fact that the integrand of the functional F satisfies the weakened Glonti–Purtukhia conditions, the functional F itself does not satisfy the weakened condition. Indeed, in this case the conditional mathematical expectation is not stochastically smooth, because we have:

$$E[\int_0^T I_{\{B_s^+\leq C\}}\mathrm{d}s|\Im_t] = \int_0^t I_{\{B_s^+\leq C\}}\mathrm{d}s + \int_t^T E[I_{\{B_s^+\leq C\}}|\Im_t]\mathrm{d}s,$$

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable[6], but the second summand is differentiable in Malliavin sense[7].

Theorem 5. *For any non-negative constant $C \geq 0$ the Brownian functional $F = \int_0^T I_{\{B_t^+ \leq C\}} \mathrm{d}t$ admits the following stochastic integral representation*

$$\int_0^T I_{\{B_t^+ \leq C\}} \mathrm{d}t = \int_0^T \Phi\left(\frac{C}{\sqrt{t}}\right) \mathrm{d}t - \int_0^T \int_t^T \frac{1}{\sqrt{s-t}} \varphi\left(\frac{C-B_t}{\sqrt{s-t}}\right) \mathrm{d}s \mathrm{d}B_t.$$

Proof. It is clear that

$$E\Big[\int_0^T I_{\{B_t^+ \leq C\}} \mathrm{d}t\Big] = \int_0^T E\big[I_{\{B_t \leq C\}}\big] \mathrm{d}t =$$

$$= \int_0^T P\{B_t/\sqrt{t} \leq C/\sqrt{t}\} \mathrm{d}t = \int_0^T \Phi(C/\sqrt{t}) \mathrm{d}t. \tag{1}$$

Next, we introduce the following notation

$$V(t,x) := E\Big[\int_t^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | B_t = x\Big]. \tag{2}$$

According to the Remark 1, using the well-known properties of conditional mathematical expectation and Brownian motion, we can write

$$V(t,x) = \{E[\int_t^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | B_t]\}|_{B_t = x} =$$

$$= \{\int_t^T E[I_{\{B_s^+ \leq C\}} | B_t] \mathrm{d}s\}|_{B_t=x} = \{\int_t^T E[I_{\{B_s \leq C\}} | \Im_t] \mathrm{d}s\}|_{B_t=x} =$$

$$= \{\int_t^T \big[\int_{-\infty}^{\infty} I_{\{y \leq C\}} \frac{1}{\sqrt{2\pi(s-t)}} \exp\{-\frac{(B_t-y)^2}{2(s-t)}\} \mathrm{d}y\big] \mathrm{d}s\}|_{B_t=x} =$$

$$= \int_t^T \{\frac{1}{\sqrt{2\pi(s-t)}} \big[\int_{-\infty}^{C} \exp\{-\frac{(x-y)^2}{2(s-t)}\} \mathrm{d}y\big]\} \mathrm{d}s =$$

$$= \int_t^T \{\frac{1}{\sqrt{2\pi}} \big[\int_{-\infty}^{\frac{C-x}{\sqrt{s-t}}} \exp\{-\frac{z^2}{2}\} \mathrm{d}z\big]\} \mathrm{d}s = \int_t^T \Phi(\frac{C-x}{\sqrt{s-t}}) \mathrm{d}s. \tag{3}$$

The last relation shows that, on the one hand, $V(t,x)$ with respect to x is an integral that depends on the parameter x. Therefore, it is not difficult to

[6] If $u_s(\omega)$ is not differentiable in Malliavin sense, then the Lebesgue average (with respect to ds) also is not differentiable in Malliavin sense (see, Theorem 2 [12]).

[7] It is well-known that if $u_s \in D_{2,1}$ for all s, then $\int_0^T u_s \mathrm{d}s \in D_{2,1}$ and

$$D_t\{\int_0^T u_s \mathrm{d}s\} = \int_0^T D_t u_s \mathrm{d}s.$$

check that in our case $V(t,x)$ is twice continuously differentiable with respect to x and we have

$$V_x^{'}(t,x) = -\int_t^T \frac{1}{\sqrt{s-t}}\varphi\left(\frac{C-x}{\sqrt{s-t}}\right)\mathrm{d}s, \tag{4}$$

$$V_{xx}^{''}(t,x) = -\int_t^T \frac{C-x}{\sqrt{(s-t)^3}}\varphi\left(\frac{C-x}{\sqrt{s-t}}\right)\mathrm{d}s.$$

On the other hand, $V(t,x)$ is an integral with variable boundary with respect to t. Now let us verify that it is continuously differentiable with respect to t. We can write

$$V_t^{'}(t,x) := \lim_{\Delta t\to 0}\frac{V(t+\Delta t,x)-V(t,x)}{\Delta t} =$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\int_{t+\Delta t}^T \Phi\Big(\frac{C-x}{\sqrt{s-t-\Delta t}}\Big)\mathrm{d}s - \int_t^T \Phi\Big(\frac{C-x}{\sqrt{s-t}}\Big)\mathrm{d}s\Big] =$$

$$= -\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\Phi\Big(\frac{C-x}{\sqrt{s-t-\Delta t}}\Big)\mathrm{d}s +$$

$$+\int_t^T \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\Phi\Big(\frac{C-x}{\sqrt{s-t-\Delta t}}\Big) - \Phi\Big(\frac{C-x}{\sqrt{s-t}}\Big)\Big]\mathrm{d}s =$$

$$= -\lim_{s\downarrow t}\Phi\Big(\frac{C-x}{\sqrt{s-t}}\Big) + \int_t^T \Phi_t^{'}\Big(\frac{C-x}{\sqrt{s-t}}\Big)\mathrm{d}s =$$

$$= -I_{\{x<C\}} - \frac{1}{2}I_{\{x=C\}} + \int_t^T \frac{C-x}{2\sqrt{(s-t)^3}}\varphi\Big(\frac{C-x}{\sqrt{s-t}}\Big)\mathrm{d}s.$$

Thus, $V(t,x)$ satisfies the conditions of the Ito's formula. Hence, according to Ito's formula, we have

$$V(t,B_t) = V(0,B_0) + \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2}V_{xx}^{''}(s,B_s)]\mathrm{d}s +$$

$$+\int_0^t V_x^{'}(s,B_s)\mathrm{d}B_s \quad (P-a.s.). \tag{5}$$

Further, from relations (1) and (2) we have

$$V(0,B_0) = E[\int_0^T I_{\{B_s^+\le C\}}\mathrm{d}s|B_0] = E[\int_0^T I_{\{B_s^+\le C\}}\mathrm{d}s|\Im_0] =$$

$$= E[\int_0^T I_{\{B_s^+\le C\}}\mathrm{d}s] = \int_0^T \Phi(C/\sqrt{t})\mathrm{d}t \quad (P-a.s.) \tag{6}$$

Meanwhile, using relation (4), we see that

$$V_x^{'}(t,B_t) = V_x^{'}(t,x)|_{x=B_t} = -\int_t^T \frac{1}{\sqrt{s-t}}\varphi\Big(\frac{C-B_t}{\sqrt{s-t}}\Big)\mathrm{d}s \quad (P-a.s.). \tag{7}$$

On the other hand, due to the Markov property of the Brownian motion

$$V(t,B_t) = \{E[\int_t^T I_{\{B_s^+\le C\}}\mathrm{d}s|B_t = x]\}|_{x=B_t} =$$

$$= E[\int_t^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | B_t] = E[\int_t^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | \Im_t] \quad (P - a.s.)$$

Hence, the process

$$\int_0^t I_{\{B_s^+ \leq C\}} \mathrm{d}s + V(t, B_t) = E[\int_0^t I_{\{B_s^+ \leq C\}} \mathrm{d}s | \Im_t] +$$

$$+ E[\int_t^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | \Im_t] = E\big[\int_0^T I_{\{B_s^+ \leq C\}} \mathrm{d}s | \Im_t]\big] := M_t \tag{8}$$

is a martingale. More over, we have

$$M_0 = V(0.B_0) \quad and \quad M_T = \int_0^T I_{\{B_s^+ \leq C\}} \mathrm{d}s. \tag{9}$$

Further, according to Levy's theorem, it is obvious that M_t is a continuous martingale. On the other hand, a continuous martingale of bounded variation starting from 0 is identically equal to 0. Hence, the part of the bounded variation of the martingale M, which is the sum of the term of bounded variation from equality (5) and the integral

$$\int_0^t I_{\{B_s^+ \leq C\}} \mathrm{d}s,$$

is equal to zero.

Therefore, taking into account relations (5)-(9), we complete the proof of the theorem. □

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