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OPTIMAL METHODS FOR RECOVERING MIXED DERIVATIVES OF NON-PERIODIC FUNCTIONS

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ОПТИМАЛЬНІ МЕТОДИ ВІДНОВЛЕННЯ МІШАНИХ ПОХІДНИХ НЕПЕРІОДИЧНИХ ФУНКЦІЙ

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Abstract. The problem of numerical differentiation for non-periodic bivariate functions is investigated. For the recovering mixed derivatives of such functions an approach on the base of truncation method is proposed. The constructed algorithms deal with Legendere polynomials, the degree of which is chosen so as to minimize the approximation error. It is established that these algorithms are order-optimal both in terms of accuracy and in the sense of the amount of Galerkin information involved.
Keywords: numerical differentiation, Legendre polynomials, truncation method, minimal radius of Galerkin information.

Анотація. Досліджується задача чисельного диференціювання неперіодичних функцій двох змінних. Для відновлення мішаних похідних таких функцій пропонується підхід, що базується на ідеї спектральної зрізки. В межах цього підходу будується поліном Лежандра, ступінь якого обирається так, щоб мінімізувати похибку наближення. Встановлено, що побудовані в такий спосіб алгоритми чисельного диференціювання є оптимальними як у сенсі точності, так і обсягу задіяної гальоркінської інформації.
Ключові слова: чисельне диференціювання, поліном Лежандра, спектральна зрізка, мінімальний радіус гальоркінської інформації.

1. Description of the problem

The problem of numerical differentiation is an actual problem arising in many applied fields such as finance, mathematical physics, image processing, analytical chemistry, viscous elastic mechanics, reliability analysis, pattern recognition and many others. The numerical differentiation is a classic problem that is unstable to small perturbations and therefore requires application of regularization to ensure the stability of the approximation. It should be noted that intensive and effective research of the stable differentiation began in the 60s of last century due to the development of the theory of ill-posed problems. The first paper on the numerical differentiation, which was written in terms of the theory of ill-posed problems, is [1]. Thus far, many researchers have proposed and substantiated different methods of numerical differentiation of univariate functions (see, for example, [2–10,12]). As to the functions of several (even two) variables, the problem is still under studying (see, in particular, [11–15]).

Let $\{\varphi_k(t)\}_{k=0}^{\infty}$ be the system of Legendre polynomials orthonormal on $[-1,1]$. By $L_2 = L_2(Q)$ we mean space of square-summable on $Q = [-1,1]^2$ functions $f(t,\tau)$ with standard inner product and standard norm.

We introduce the space of smooth functions

$$L^{\mu}_{2,2}(Q) = \{f \in L_2(Q): \quad \|f\|^2_{\mu} = \sum_{k,j=0}^{\infty} (\underline{k\cdot j})^{2\mu} |\langle f, \varphi_{k,j}\rangle|^2 < \infty\}, \quad \mu > 0, \tag{1}$$

where $\langle f, \varphi_{k,j}\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)\varphi_k(t)\varphi_j(\tau)d\tau dt, \quad k,j = 0,1,2,\ldots,$ are Fourier-Legendre coefficients of f, $\underline{k} = \max\{1,k\}$. Note that in the future we will use the same notations both for space and for a unite ball from this space: $L^{\mu}_{2,2} = L^{\mu}_{2,2}(Q) = \{f \in L^{\mu}_{2,2} : \|f\|_{\mu} \le 1\}$, what we call a class of functions. What exactly is meant by $L^{\mu}_{2,2}$, space or class, will be clear depending on the context in each case.

It should be noted that $L^{\mu}_{2,2}$ is generalization of the class of bivariate functions with dominating mixed derivatives. Moreover, let $C = C(Q)$ be the space of continuous bivariate functions on Q .

We represent a function $f(t,\tau)$ from $L^{\mu}_{2,2}$, $\mu > 4$, as

$$f(t,\tau) = \sum_{k,j=0}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi_k(t)\varphi_j(\tau), \tag{2}$$

and by its mixed derivative $f^{(2,2)}$ we mean the following series

$$f^{(2,2)}(t,\tau) = \sum_{k,j=2}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi''_k(t)\varphi''_j(\tau). \tag{3}$$

Assume that instead of the exact values of the Fourier-Legendre coefficients $\langle f, \varphi_{k,j}\rangle$ only some their perturbations are known with the error level δ in the metric of ℓ_p, $1 \le p \le \infty$. More accurately, we assume that there is a sequence of numbers $\overline{f^{\delta}} = \{\langle f^{\delta}, \varphi_{k,j}\rangle\}_{k,j\in\mathbb{N}_0}$ such that for $\overline{\xi} = \{\xi_{k,j}\}_{k,j\in\mathbb{N}_0}$, where $\xi_{k,j} = \langle f - f^\delta, \varphi_{k,j}\rangle$, and for some $1 \le p \le \infty$ the relation

$$\|\overline{\xi}\|_{\ell_p} \le \delta, \quad 0 < \delta < 1, \tag{4}$$

is true.

The research of this work is devoted to the optimization of methods for recovering the derivative (3) of functions from the class $L^\mu_{2,2}$. Further, we give a strict statement of the problem to be studied. In the coordinate plane $[2,\infty) \times [2,\infty)$ we take an arbitrary bounded area Ω. By $\mathrm{card}(\Omega)$ we mean the number of points that make up Ω and by the information vector $G(\Omega, \overline{f}^\delta) \in \mathbb{R}^N$, $\mathrm{card}(\Omega) = N$, we take the set of perturbed values of Fourier-Legendre coefficients $\{\langle f^\delta, \varphi_{k,j}\rangle\}_{(k,j)\in\Omega}$.

Let $X = L_2(Q)$ or $X = C(Q)$. By numerical differentiation algorithm we mean any mapping $\psi^{(2,2)} = \psi^{(2,2)}(\Omega)$ that corresponds to the information vector $G(\Omega, \overline{f}^\delta)$ an element $\psi^{(2,2)}(G(\Omega, \overline{f}^\delta)) \in X$, which is taken as an approximation to the derivative (3) of function f from class $L^\mu_{2,2}$. We denote by $\Psi(\Omega)$ the set of all algorithms $\psi^{(2,2)}(\Omega) : \mathbb{R}^N \to X$, that use the same information vector $G(\Omega, \overline{f}^\delta)$.

Actually, we do not require from algorithms from $\Psi(\Omega)$, in generally speaking, either linearity or even stability . The only condition for algorithms from $\Psi(\Omega)$ is to use input information in the form of perturbed values of the Fourier-Legendre coefficients with indices from the domain Ω of the coordinate plane. Such a general understanding of the algorithm is explained by the desire to compare the widest possible range of possible methods of numerical differentiation.

The error of the algorithm $\psi^{(2,2)}$ on the class $L^\mu_{2,2}$ is determined by the quantity

$$\varepsilon_\delta(L^\mu_{2,2}, \psi^{(2,2)}(\Omega), X, \ell_p) = \sup_{f\in L^\mu_{2,2},\, \|f\|_\mu \le 1} \ \sup_{\overline{f}^\delta :\, (4)} \|f^{(2,2)} - \psi^{(2,2)}(G(\Omega, \overline{f}^\delta))\|_X.$$

The minimal radius of the Galerkin information for the problem of numerical differentiation on the class $L^\mu_{2,2}$ is given by

$$R^{(2,2)}_{N,\delta}(L^\mu_{2,2}, X, \ell_p) = \inf_{\Omega:\, \mathrm{card}(\Omega)\le N} \ \inf_{\psi^{(2,2)}\in\Psi(\Omega)} \varepsilon_\delta(L^\mu_{2,2}, \psi^{(2,2)}(\Omega), X, \ell_p).$$

The value $R^{(2,2)}_{N,\delta}(L^\mu_{2,2}, X, \ell_p)$ describes the minimal possible accuracy in the metric of space X, which can be achieved by numerical differentiation of arbitrary function $f \in L^\mu_{2,2}$, while using not more than N values of its Fourier-Legendre coefficients that are δ-perturbed in the ℓ_p metric. Note that the minimal radius of Galerkin information in the problem of recovering the first partial derivative was studied in [16], and for other types of ill-posed problems, similar studies were previously carried out in [17, 18]. It should be added that the minimal radius characterizes the information complexity of the considered problem and is traditionally studied within the framework of the IBC-theory (Information Based Complexity Theory), the foundations of which are laid in monographs [19] and [20].

The aim of our research is to construct numerical differentiation methods that are optimal in terms of the quantities $R_{N,\delta}^{(2,2)}(L_{2,2}^{\mu}, C, \ell_p)$ and $R_{N,\delta}^{(2,2)}(L_{2,2}^{\mu}, L_2, \ell_p)$.

2. Truncation method. Error estimate in L_2 metric

It should be noted that at the moment a number of approaches were developed for numerical differentiation (see, for example, [11, 21–23], and also see [24] and its bibliography). All these methods are accepted to divide into three groups (see [11]): difference methods, interpolation methods and regularization methods. As is known, the first two types of methods have their advantage in the simplicity of implementation, but they guarantee satisfactory accuracy only in the case of exactly given input data about the differentiable function. In the same time regularization methods give stable approximations to the desired derivatives in the case of perturbed input data but most of them (for example, the Tikhonov method and its various variations) are quite complicated for numerical realization in view of their integral form and require hard-to-implement rules for determination of regularization parameters (see [11]). Recently in [15] a concise numerical method, called the truncation method, has been proposed as stable and simple approach to numerical differentiation of multivariable functions. The essence of this method is to replace the Fourier series (3) with a finite Fourier sum using perturbed data $\langle f^\delta, \varphi_{k,j}\rangle$. In the truncation method to ensure the stability of the approximation and achieve the required order accuracy, it is necessary to choose properly the discretization parameter, which here serves as a regularization parameter. So, the process of regularization in method under consideration consists in matching the discretization parameter with the perturbation level of the input data. Simplicity of implementation is the main advantage of this method.

In the case of an arbitrary bounded domain Ω of the coordinate plane $[2,\infty)\times[2,\infty)$, the truncation method for differentiating functions of two variables has the form

$$\mathcal{D}_\Omega f^\delta(t,\tau) = \sum_{(k,j)\in\Omega} \langle f^\delta, \varphi_{k,j}\rangle \varphi_k''(t)\varphi_j''(\tau).$$

In order to increase the efficiency of the approach under study, we take the hyperbolic cross as the domain Ω of the following form

$$\Omega = \Gamma_n := \{(k,j) : \ k{\cdot}j \le 2n{-}1, \quad k,j = 2,\dots,n{-}1\}, \qquad \mathrm{card}(\Gamma_n) = O(n\ln n).$$

Then the version of proposed truncation method can be written as

$$\overline{\mathcal{D}}_n f^\delta(t,\tau) = \sum_{k,j\ge 2,\ kj\le 2n-1} \langle f^\delta, \varphi_{k,j}\rangle \varphi_k''(t)\varphi_j''(\tau). \tag{5}$$

We note that earlier the idea of a hyperbolic cross for the problem of numerical differentiation was used in the papers [15, 16, 25] (for more details about usage of hyperbolic cross in solving the other ill-posed problems see [26–29]).

The approximation properties of the method (5) will be investigated in Sections 2 and 3 while in Section 4 it will be established that the method (5) is order-optimal in the sense of the minimal radius of the Galerkin information.

Let us write the error of the method (5) as

$$f^{(2,2)}(t,\tau)-\overline{\mathcal{D}}_n f^\delta(t,\tau) = \Big(f^{(2,2)}(t,\tau) - \overline{\mathcal{D}}_n f(t,\tau)\Big)+\Big(\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f^\delta(t,\tau)\Big). \tag{6}$$

For our calculations, we need the following formula (see Lemma 18 [30])

$$\varphi_k'(t) = 2\sqrt{k+1/2}\sum_{l=0}^{k-1}{}^{*}\sqrt{l+1/2}\,\varphi_l(t), \quad k\in\mathbb{N}, \tag{7}$$

where in aggregate $\sum_{l=0}^{k-1}{}^{*}\sqrt{l+1/2}\,\varphi_l(t)$ the summation is extended over only those terms for which $k+l$ is odd.

Let us estimate the error of the method (5) in the metric of L_2. A upper bound for difference (**??**) is contained in the following statement.

Lemma 1. *Let $f\in L^\mu_{2,2}$, $\mu>4$. Then*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f\|_{L_2} \le c\|f\|_\mu n^{-\mu+4}\ln n.$$

The following statement contains an estimate for the second difference from the right-hand side of (6) in the metric of L_2.

Lemma 2. *Let the condition (4) be satisfied. Then for arbitrary function $f\in L_2(Q)$ it holds true*

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f^\delta\|_{L_2} \le c\delta n^{9/2-1/p}\ln^{3/2-1/p} n.$$

The combination of Lemmas 1 and 2 gives

Theorem 1. *Let $f\in L^\mu_{2,2}$, $\mu>4$. Then for $n\asymp\left(\delta^{-1}\ln^{1/p-1/2}\frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ it holds*

$$\|f^{(2,2)} - \bar{\mathcal{D}}_n f^\delta\|_{L_2} \le c\left(\delta\ln^{1/2-1/p}\frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}}\ln\frac{1}{\delta}.$$

3. Truncation method. Error estimate in the C metric

Now we have to estimate the error of (5) in the metric of C.

Lemma 3. *Let $f\in L^\mu_{2,2}$, $\mu>5$. Then we have*

$$\|f^{(2,2)} - \overline{\mathcal{D}}_n f\|_C \le c\|f\|_\mu n^{-\mu+5}\ln^{3/2} n.$$

The following statement contains an estimate for the second difference from the right-hand side of (6) in the metric of C.

Lemma 4. *Let the condition (4) be satisfied. Then for arbitrary function $f\in C(Q)$ it holds true*

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f^\delta\|_C \le c\delta n^{11/2-1/p}\ln^{2-1/p} n.$$

The combination of Lemmas 3 and 4 gives

Theorem 2. *Let $f \in L^{\mu}_{2,2}$, $\mu > 5$. Then for $n \asymp \left(\delta^{-1} \ln^{1/p-1/2} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ it holds*

$$\|f^{(2,2)} - \bar{\mathcal{D}}_n f^{\delta}\|_C \le c \left(\delta \ln^{1/2-1/p} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \ln^{3/2} \frac{1}{\delta}.$$

4. Minimal radius of Galerkin information

First, estimates by the order of the minimal radius of Galerkin information in the metric of C will be established.

Theorem 3. *Let $\mu > 5$, $1 \le p \le \infty$, $N \ge \left(2^{\mu}\delta/\widetilde{c}\right)^{-1/(\mu+1/2-1/p)}$. Then*

$$R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p) \ge \overline{c}\, N^{-\mu+5}.$$

Theorem 4. *Let $\mu > 5$, $1 \le p \le \infty$. Then for $N \asymp \left(\delta^{-1} \ln^{\mu} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ it holds*

$$N^{-\mu+5} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p) \preceq N^{-\mu+5} \ln^{\mu-7/2} N,$$

$$\left(\delta \ln^{-\mu} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, C, \ell_p) \preceq \left(\delta \ln^{1/2-1/p} \frac{1}{\delta}\right)^{\frac{\mu-5}{\mu-1/p+1/2}} \ln^{3/2} \frac{1}{\delta}.$$

The upper bound is implemented by (5) for $n \asymp \left(\delta^{-1} \ln^{1/p-1/2} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$.

Bounds for the minimal radius of Galerkin information in the metric of L_2 are contained in the following assertions.

Theorem 5. *Let $\mu > 4$, $1 \le p \le \infty$. Then for $N \ge \left(2^{\mu}\delta/\widetilde{c}\right)^{-1/(\mu+1/2-1/p)}$ it holds*

$$R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \ge \overline{\overline{c}}\, N^{-\mu+4}, \qquad \overline{\overline{c}} = \frac{3\sqrt{15}}{2^{\mu+3}}\widetilde{c}.$$

Theorem 6. *Let $\mu > 4$, $1 \le p \le \infty$. Then for $N \asymp \left(\delta^{-1} \ln^{\mu} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$ it holds*

$$N^{-\mu+4} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \preceq N^{-\mu+4} \ln^{\mu-3} N,$$

$$\left(\delta \ln^{-\mu} \frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}} \preceq R^{(2,2)}_{N,\delta}(L^{\mu}_{2,2}, L_2, \ell_p) \preceq \left(\delta \ln^{1/2-1/p} \frac{1}{\delta}\right)^{\frac{\mu-4}{\mu-1/p+1/2}} \ln \frac{1}{\delta}.$$

The upper bound is implemented by (5) for $n \asymp \left(\delta^{-1} \ln^{1/p-1/2} \frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/2}}$.

5. Conclusion

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