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ПРО КОНЗИСТЕНТНІСТЬ ОЦІНКИ ІМПУЛЬСНОЇ ПЕРЕХІДНОЇ ФУНКЦІЇ

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ON CONSISTENCY FOR THE ESTIMATOR OF IMPULSE RESPONSE FUNCTION

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ABSTRACT. The aim of the article is to construct and analyze a statistical estimator for the real-valued impulse response function (IRF) of a time-invariant continuous linear system, and to establish its properties of asymptotic unbiasedness and consistency. The estimation of the response function is performed using a sample input-output cross-correlogram approach. The input signal is modeled as a stationary zero-mean Gaussian stochastic process, represented as a trimmed Fourier series using a trigonometric orthonormal basis. The study employs functional analysis and probability theory to derive upper bounds for the mathematical expectation and variance of the proposed estimator. The study establishes that the proposed integral cross-correlogram estimator is asymptotically unbiased as the cutoff level N approaches infinity. Furthermore, by evaluating the upper bounds of the bias and variance, it is proven that the estimator is consistent in the mean square sense as both the cutoff level N and the averaging interval length T approach infinity. The proposed approach

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offers a rigorous mathematical framework for system identification applicable to signal processing, automatic control, econometrics, and oceanology. The use of a Fourier series representation makes this method particularly effective for analyzing linear systems driven by periodic or quasi-periodic input signals.

KEYWORDS: impulse response function, cross-correlogram, consistency, Fourier series.

АНОТАЦІЯ. Метою статті є побудова та аналіз статистичної оцінки для дійсної імпульсної перехідної функції (ПФ) часо-інваріантної неперервної лінійної системи, а також доведення її асимптотичної незсушеності та слухності (конзистентності). Оцінювання перехідної функції здійснюється за допомогою методу крос-корелограми вхід-вихід. Вхідний сигнал моделюється як стаціонарний гауссівський випадковий процес із нульовим середнім, представлений зрізаним рядом Фур'є за тригонометричним ортонормованим базисом. У роботі застосовано методи функціонального аналізу та теорії ймовірностей для виведення верхніх оцінок математичного сподівання та дисперсії. Встановлено, що запропонована інтегральна оцінка крос-корелограми є асимптотично незсушеною при прямуванні рівня зрізання N до нескінченності. Крім того, на основі отриманих верхніх оцінок зміщення та дисперсії доведено, що оцінка є слухною у середньому квадратичному сенсі при одночасному прямуванні N та довжини інтервалу усереднення T до нескінченності. Запропонований підхід створює строгу математичну базу для ідентифікації систем, що може бути застосована в обробці сигналів, автоматичному керуванні, економітриці та океанології. Використання представлення у вигляді ряду Фур'є робить цей метод особливо ефективним для аналізу систем із періодичними або квазіперіодичними вхідними сигналами.

КЛЮЧОВІ СЛОВА: імпульсна перехідна функція, крос-корелограма, конзистентність, ряд Фур'є.

1. INTRODUCTION

The estimation of stochastic linear systems driven by impulse response functions (IRFs) has been a prominent area of research with applications spanning signal processing, automatic control, econometrics, and oceanology. Various statistical and deterministic methods have been employed to estimate the IRF from observations of a single-input single-output (SISO) system, including perturbing the system with stationary stochastic processes [1].

Buldygin and colleagues developed methods for estimating the unknown impulse response function of linear systems relying on the construction of a sample cross-correlogram between the input stochastic process and the system's response [2], [3], [4]. Further developments for integral-type cross-correlogram estimators, including supremum error inequalities, new estimator construction methods, and hypothesis testing, were explored in [5], [6], [7], [8].

This paper considers a time-invariant continuous linear system with a real-valued impulse response function defined on a bounded domain. The estimator of the response function is based on a single observation of the input-output cross-correlogram. The input processes are assumed to be zero-mean stationary Gaussian processes, represented as trimmed Fourier series. The estimator depends on the averaging interval length T for the cross-correlogram and the cutoff level N for the input signal. The asymptotic unbiasedness of the estimator as $N \rightarrow \infty$ and its consistency as $N \rightarrow \infty$ and $T \rightarrow \infty$ are investigated.

The paper is organized as follows. Section 2 introduces the main definitions, describes the input process using a trigonometric orthonormal basis, derives upper bounds for the mathematical expectation and variance of the cross-correlogram estimator, and establishes its asymptotic unbiasedness and consistency. Section 3 contains concluding remarks.

2. ESTIMATING THE IMPULSE RESPONSE FUNCTION AND ITS CONSISTENCY

Consider a time-invariant continuous linear system characterized by a real-valued, square-integrable impulse response function (IRF) $H(\tau)$, defined on a finite domain $\tau \in [0, \Lambda]$. This implies that the system's response $Y(t)$ to an input signal $X(t)$, where $t \in \mathbb{R}$, is given by:

$$Y(t) = \int_0^\Lambda H(\tau)X(t - \tau) d\tau, \quad t \in \mathbb{R}, \quad (1)$$

with $H \in L_2([0, \Lambda])$. A key problem in linear system theory is estimating the function H from observations of the system's responses to specific input signals. In this paper, we employ a cross-correlogram approach to estimate the unknown IRF H . First, we describe the input process, which is constructed using a trigonometric orthonormal basis.

Let the following system of functions be a trigonometric orthonormal basis in $L_2([0, \Lambda])$:

$$\left\{ \frac{1}{\sqrt{\Lambda}}, \sqrt{\frac{2}{\Lambda}} \cos\left(\frac{2k\pi t}{\Lambda}\right), \sqrt{\frac{2}{\Lambda}} \sin\left(\frac{2k\pi t}{\Lambda}\right), k \geq 1 \right\}. \quad (2)$$

Now, consider as the input to the linear system a real-valued, Gaussian, stationary, zero-mean stochastic process $X = X_N = (X_N(u), u \in \mathbb{R})$, which can be expressed as:

$$X_N(u) = \sqrt{\frac{2}{\Lambda}} \sum_{k=1}^N \left(\xi_k \cos\left(\frac{2k\pi u}{\Lambda}\right) + \eta_k \sin\left(\frac{2k\pi u}{\Lambda}\right) \right), \quad u \in \mathbb{R}, \quad (3)$$

where $N > 0$ is a fixed integer, and ξ_k and η_k are uncorrelated Gaussian random variables for $k \geq 0$, satisfying $\mathbf{E}\xi_k = \mathbf{E}\eta_k = 0$ and $\mathbf{E}\xi_k^2 = \mathbf{E}\eta_k^2 = 1$.

Remark 1. The input signal $X_N(t)$ can be viewed as a model of a stochastic process that can be expanded into a series over a given orthonormal basis. For further details on constructing such models with specified accuracy and reliability, we refer readers to [9], [10], [11], [12], [13], [14], and [15].

Let us find the covariance function of the stationary process X_N . Using the properties of the random variables ξ_k and η_k , we have:

$$\begin{aligned} \mathbf{E}X_N(t)X_N(s) &= \mathbf{E}\left[\frac{2}{\Lambda}\left(\sum_{k=1}^N \xi_k \cos\left(\frac{2k\pi t}{\Lambda}\right) + \sum_{k=1}^N \eta_k \sin\left(\frac{2k\pi t}{\Lambda}\right)\right)\right. \\ &\quad \left.\times \left(\sum_{j=1}^N \xi_j \cos\left(\frac{2j\pi s}{\Lambda}\right) + \sum_{j=1}^N \eta_j \sin\left(\frac{2j\pi s}{\Lambda}\right)\right)\right] \\ &= \frac{2}{\Lambda} \sum_{k=1}^N \left(\cos\left(\frac{2k\pi t}{\Lambda}\right) \cos\left(\frac{2k\pi s}{\Lambda}\right) + \sin\left(\frac{2k\pi t}{\Lambda}\right) \sin\left(\frac{2k\pi s}{\Lambda}\right)\right) \\ &= \frac{2}{\Lambda} \sum_{k=1}^N \cos\left(\frac{2k\pi(t-s)}{\Lambda}\right). \end{aligned}$$

Thus, the covariance function $r_N(t-s) = \mathbf{E}X_N(t)X_N(s)$ is given by:

$$r_N(t-s) = \frac{2}{\Lambda} \sum_{k=1}^N \cos\left(\frac{2k\pi(t-s)}{\Lambda}\right). \quad (4)$$

When the system (1) is perturbed by the stochastic process X_N , the corresponding output process is:

$$Y_N(t) = \int_0^\Lambda H(\tau)X_N(t-\tau) d\tau.$$

Let a_0 represent the system's output for a constant signal, defined as:

$$a_0 = \frac{1}{\sqrt{\Lambda}} \int_0^\Lambda H(t) dt. \quad (5)$$

Define:

$$H^*(\tau) = H(\tau) - a_0. \quad (6)$$

As an estimator for the difference $H^*(\tau)$ between the IRF and a_0 , we consider the integral cross-correlogram:

$$\hat{H}(\tau) = \hat{H}_{N,T,\Lambda}(\tau) = \frac{1}{T} \int_0^T Y_N(t)X_N(t-\tau) dt, \quad (7)$$

where $T > 0$ is the averaging parameter.

Remark 2. The integral in (1) is interpreted as a mean-square Riemann integral. This integral exists if and only if the following Riemann integral exists (see [16]):

$$\int_0^\Lambda \int_0^\Lambda H(\tau)r_N(s-\tau)H(s) ds d\tau. \quad (8)$$

The covariance function of the process X_N is given by (4). Since the trigonometric functions (2) form an orthonormal basis in $L_2([0, \Lambda])$ and $H \in L_2([0, \Lambda])$, the integral (8) exists. Consequently, the integral in (1) also exists.

Now, define:

$$a_k = \sqrt{\frac{2}{\Lambda}} \int_0^\Lambda H(t) \cos\left(\frac{2k\pi t}{\Lambda}\right) dt, \quad b_k = \sqrt{\frac{2}{\Lambda}} \int_0^\Lambda H(t) \sin\left(\frac{2k\pi t}{\Lambda}\right) dt. \quad (9)$$

Since $H \in L_2([0, \Lambda])$, the function H can be expanded into a Fourier series using the trigonometric orthonormal basis (2) on the interval $[0, \Lambda]$. This yields:

$$H(t) = \frac{a_0}{\sqrt{\Lambda}} + \sqrt{\frac{2}{\Lambda}} \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2k\pi t}{\Lambda}\right) + b_k \sin\left(\frac{2k\pi t}{\Lambda}\right) \right), \quad (10)$$

where a_k and b_k are as defined in (9).

Remark 3. By the Riesz–Fischer theorem (see [17]), the series (10) converges in the mean squared sense. If $H(t)$ and its derivative $H'(t)$ are continuous, the convergence in (10) can also be considered in the pointwise and uniform senses.

Lemma 1. *The following relations hold:*

$$\begin{aligned} \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) &= \\ &= \sqrt{\frac{2}{\Lambda}} \sum_{k=1}^N \left(a_k \cos\left(\frac{2k\pi\tau}{\Lambda}\right) + b_k \sin\left(\frac{2k\pi\tau}{\Lambda}\right) \right), \quad \tau \in [0, \Lambda], \end{aligned} \quad (11)$$

and

$$\begin{aligned} H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) &= \\ &= \sqrt{\frac{2}{\Lambda}} \sum_{k=N+1}^{\infty} \left(a_k \cos\left(\frac{2k\pi\tau}{\Lambda}\right) + b_k \sin\left(\frac{2k\pi\tau}{\Lambda}\right) \right), \quad \tau \in [0, \Lambda]. \end{aligned} \quad (12)$$

Proof. It is obvious that the estimator $\hat{H}_{N,T,\Lambda}(\tau)$ is biased in the general case. Indeed,

$$\mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) = \frac{1}{T} \int_0^T \mathbf{E}[Y_N(t)X_N(t-\tau)] dt = \int_0^\Lambda H(v)r_N(v-\tau) dv. \quad (13)$$

Substituting in (13) the values from (10) and (4), we have

$$\begin{aligned} \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) &= \frac{2}{\Lambda} \int_0^\Lambda H(v) \sum_{l=1}^N \left(\cos\left(\frac{2l\pi v}{\Lambda}\right) \cos\left(\frac{2l\pi\tau}{\Lambda}\right) \right. \\ &\quad \left. + \sin\left(\frac{2l\pi v}{\Lambda}\right) \sin\left(\frac{2l\pi\tau}{\Lambda}\right) \right) dv \\ &= \frac{2}{\Lambda} \sum_{l=1}^N \left(\cos\left(\frac{2l\pi\tau}{\Lambda}\right) \int_0^\Lambda H(v) \cos\left(\frac{2l\pi v}{\Lambda}\right) dv \right. \\ &\quad \left. + \sin\left(\frac{2l\pi\tau}{\Lambda}\right) \int_0^\Lambda H(v) \sin\left(\frac{2l\pi v}{\Lambda}\right) dv \right) \\ &= \sqrt{\frac{2}{\Lambda}} \sum_{l=1}^N \left(a_l \cos\left(\frac{2l\pi\tau}{\Lambda}\right) + b_l \sin\left(\frac{2l\pi\tau}{\Lambda}\right) \right). \end{aligned}$$

Let $\varphi_k(\tau) = \sqrt{\frac{2}{\Lambda}} \cos\left(\frac{2k\pi\tau}{\Lambda}\right)$ and $\psi_k(\tau) = \sqrt{\frac{2}{\Lambda}} \sin\left(\frac{2k\pi\tau}{\Lambda}\right)$. Then,

$$\mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) = \sum_{k=1}^N (\varphi_k(\tau)a_k + \psi_k(\tau)b_k).$$

Therefore, relation (11) is proved.

Equalities (10) and (11) imply

$$H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) = \sum_{k=N+1}^{\infty} (\varphi_k(\tau)a_k + \psi_k(\tau)b_k), \quad \tau \in [0, \Lambda].$$

The lemma is completely proved. \square

Consider the following conditions:

Condition A. The function $H(\tau)$ is twice differentiable on $[0, \Lambda]$. The functions $H(\tau)$ and $H'(\tau)$ are continuous on $[0, \Lambda]$, and:

$$\begin{aligned} I_0 &= I_0(\Lambda) = \int_0^{\Lambda} |H(\tau)| d\tau < \infty, \\ I_1 &= I_1(\Lambda) = \left(\int_0^{\Lambda} |H'(\tau)|^2 d\tau \right)^{1/2} < \infty, \\ I_2 &= I_2(\Lambda) = \int_0^{\Lambda} |H''(\tau)| d\tau < \infty. \end{aligned}$$

Condition B. The following relation holds:

$$H(0) = H(\Lambda).$$

Remark 4. Condition B implies that the effect of the impulse on the interval $[0, \Lambda]$ must fully dissipate. If this condition is not met, the graph can be rotated by an angle $\arctan \frac{H(\Lambda) - H(0)}{\Lambda}$.

Define:

$$d = |H'(0)| + |H'(\Lambda)|. \quad (14)$$

Theorem 1 (Kozachenko Yu., Rozora I., [18]). *Assume that Conditions A and B are satisfied. Then:*

$$|H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau)| \leq \frac{\Lambda(d + I_2(\Lambda))}{2\pi^2 N}, \quad (15)$$

$$\text{Var}\hat{H}_{N,T,\Lambda}(\tau) \leq \frac{3\Lambda^3 I_1^2}{\pi^4 T^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right)^2, \quad (16)$$

where $I_1(\Lambda)$ and $I_2(\Lambda)$ are defined in Condition A.

Proof. From (12) and (9) it follows that

$$\begin{aligned}
 H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) &= \sum_{k=N+1}^{\infty} (\varphi_k(\tau)a_k + \psi_k(\tau)b_k) \\
 &= \frac{2}{\Lambda} \sum_{k=N+1}^{\infty} \int_0^{\Lambda} H(u) \left(\cos\left(\frac{2k\pi u}{\Lambda}\right) \cos\left(\frac{2k\pi\tau}{\Lambda}\right) \right. \\
 &\quad \left. + \sin\left(\frac{2k\pi u}{\Lambda}\right) \sin\left(\frac{2k\pi\tau}{\Lambda}\right) \right) du \\
 &= \frac{2}{\Lambda} \sum_{k=N+1}^{\infty} \int_0^{\Lambda} H(u) \cos\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du. \quad (17)
 \end{aligned}$$

Estimate now the integral in the equality above using first partial integration and condition **B**:

$$\begin{aligned}
 &\int_0^{\Lambda} H(u) \cos\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du \\
 &= \left(H(u) \frac{\Lambda}{2k\pi} \sin\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) \Big|_0^{\Lambda} - \frac{\Lambda}{2k\pi} \int_0^{\Lambda} H'(u) \sin\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du \right) \\
 &= -\frac{\Lambda}{2k\pi} \int_0^{\Lambda} H'(u) \sin\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du \\
 &= -\frac{\Lambda}{2k\pi} \left(-H'(u) \frac{\Lambda}{2k\pi} \cos\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) \Big|_0^{\Lambda} \right. \\
 &\quad \left. + \frac{\Lambda}{2k\pi} \int_0^{\Lambda} H''(u) \cos\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du \right) \\
 &= \frac{(H'(0) \cos(2k\pi\tau/\Lambda) - \cos(2k\pi\tau/\Lambda) H'(\Lambda)) \Lambda^2}{(2k\pi)^2} \\
 &\quad - \frac{\Lambda^2}{(2k\pi)^2} \int_0^{\Lambda} H''(u) \cos\left(\frac{2k\pi(u-\tau)}{\Lambda}\right) du. \quad (18)
 \end{aligned}$$

Then by conditions **A**, **B**, notation (14) and relationship (18) we obtain

$$|\varphi_k(\tau)a_k + \psi_k(\tau)b_k| \leq \frac{2\Lambda(d + I_2)}{(2k\pi)^2} \quad (19)$$

Relations (17) and (19) imply that

$$\begin{aligned}
 |H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau)| &\leq \sum_{k=N+1}^{\infty} |\varphi_k(\tau)a_k + \psi_k(\tau)b_k| \\
 &\leq \frac{\Lambda(d + I_2)}{2\pi^2} \sum_{k=N+1}^{\infty} \frac{1}{k^2}. \quad (20)
 \end{aligned}$$

Since

$$\sum_{k=N+1}^{\infty} \frac{1}{k^2} = \sum_{k=N+1}^{\infty} \int_{k-1}^k \frac{1}{k^2} dx < \sum_{k=N+1}^{\infty} \int_{k-1}^k \frac{1}{x^2} dx = \int_N^{\infty} \frac{1}{x^2} dx = \frac{1}{N}, \quad (21)$$

then the biasness of $\hat{H}_{N,T,\Lambda}(\tau)$ to the parameter $H^*(\tau)$ can be evaluated as

$$|H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau)| \leq \frac{\Lambda(d + I_2)}{2\pi^2 N},$$

and inequality (15) is completely proved.

To estimate the variance, we proceed as follows. Denote

$$\begin{aligned} R_H(t, s) &= \int_0^\Lambda \int_0^\Lambda H(v)H(u)r_N(t-s+u-v)dudv, \\ R_1(t, s, \tau) &= \int_0^\Lambda H(v)r_N(t-s+\tau-v)dv, \\ R_2(t, s, \tau) &= \int_0^\Lambda H(u)r_N(s-t+\tau-u)du, \end{aligned} \quad (22)$$

$$\begin{aligned} &Var\hat{H}_{N,T,\Lambda}(\tau) \\ &= \frac{1}{T^2} \int_0^T \int_0^T \left[\int_0^\Lambda \int_0^\Lambda H(v)H(u)r_N(t-s+u-v)dudv \cdot r_N(t-s) \right. \\ &+ \left. \int_0^\Lambda H(v)r_N(t-s+\tau-v)dv \cdot \int_0^\Lambda H(u)r_N(s-t+\tau-u)du \right] dt ds \\ &= \frac{1}{T^2} \left(\int_0^T \int_0^T R_H(t, s)r_N(t-s)dt ds \right. \\ &\quad \left. + \int_0^T \int_0^T R_1(t, s, \tau) \cdot R_2(t, s, \tau)dt ds \right). \end{aligned} \quad (23)$$

By (4) we have

$$\begin{aligned} R_H(t, s) &= \int_0^\Lambda \int_0^\Lambda H(v)H(u)r_N(t-s+u-v)dudv \\ &= \frac{2}{\Lambda} \sum_{k=1}^N \left(\int_0^\Lambda H(u) \cos\left(\frac{2k\pi(t+u)}{\Lambda}\right) du \int_0^\Lambda H(v) \cos\left(\frac{2k\pi(s+v)}{\Lambda}\right) dv \right. \\ &\quad \left. + \int_0^\Lambda H(u) \sin\left(\frac{2k\pi(t+u)}{\Lambda}\right) du \int_0^\Lambda H(v) \sin\left(\frac{2k\pi(s+v)}{\Lambda}\right) dv \right) \\ &= \frac{2}{\Lambda} \sum_{k=1}^N (c_H(t)c_H(s) + s_H(t)s_H(s)), \end{aligned} \quad (24)$$

where

$$c_H(t) = \int_0^\Lambda H(u) \cos\left(\frac{2k\pi(t+u)}{\Lambda}\right) du, \quad s_H(t) = \int_0^\Lambda H(u) \sin\left(\frac{2k\pi(t+u)}{\Lambda}\right) du.$$

Similarly to (18) we can estimate the functions $c_H(t)$ and $s_H(t)$. Really, it follows from conditions **A**, **B**

$$\begin{aligned}
 |c_H(t)| &= \left| \int_0^\Lambda H(u) \cos\left(\frac{2k\pi(t+u)}{\Lambda}\right) du \right| \\
 &= \left| \left(H(u) \frac{\Lambda}{2k\pi} \sin\left(\frac{2k\pi(u+t)}{\Lambda}\right) \right) \Big|_0^\Lambda - \frac{\Lambda}{2k\pi} \int_0^\Lambda H'(u) \sin\left(\frac{2k\pi(u+t)}{\Lambda}\right) du \right| \\
 &= \left| \frac{\Lambda}{2k\pi} \int_0^\Lambda H'(u) \sin\left(\frac{2k\pi(u+t)}{\Lambda}\right) du \right| \\
 &\leq \frac{\Lambda}{2k\pi} \left(\int_0^\Lambda (H'(u))^2 du \right)^{\frac{1}{2}} \left(\int_0^\Lambda \sin^2\left(\frac{2k\pi(u+t)}{\Lambda}\right) du \right)^{\frac{1}{2}} \\
 &= \frac{\Lambda I_1(\Lambda)}{2k\pi} \sqrt{\frac{\Lambda}{2}}. \quad (25)
 \end{aligned}$$

The same evaluation could be made for $s_H(t)$

$$|s_H(t)| \leq \frac{\Lambda I_1(\Lambda)}{2k\pi} \sqrt{\frac{\Lambda}{2}}. \quad (26)$$

Using a similar method for estimation as in (21), it is easy to prove that

$$\sum_{k=1}^N \frac{1}{k^2} \leq \frac{\pi^2}{6} - \frac{1}{N}. \quad (27)$$

Then substituting (25) and (26) into (24) and using (27) we obtain

$$\begin{aligned}
 R_H(t, s) &\leq \frac{4}{\Lambda} \sum_{k=1}^N \left(\frac{\Lambda I_1(\Lambda)}{2k\pi} \sqrt{\frac{\Lambda}{2}} \right)^2 \\
 &= \frac{\Lambda^2 I_1^2}{2\pi^2} \sum_{k=1}^N \frac{1}{k^2} \leq \frac{\Lambda^2 I_1^2}{2\pi^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right), \quad (28)
 \end{aligned}$$

that doesn't depend on t and s . To evaluate the first summand in $Var \hat{H}_{N,T,\Lambda}(\tau)$, find now

$$\begin{aligned}
 &\frac{1}{T^2} \int_0^T \int_0^T r_N(t-s) dt ds \\
 &= \frac{2}{T^2 \Lambda} \sum_{k=1}^N \left(\left(\int_0^T \cos\left(\frac{2k\pi t}{\Lambda}\right) dt \right)^2 + \left(\int_0^T \sin\left(\frac{2k\pi t}{\Lambda}\right) dt \right)^2 \right) \\
 &= \frac{2}{T^2 \Lambda} \sum_{k=1}^N \left(\left(\frac{\Lambda}{2k\pi} \sin\left(\frac{2k\pi T}{\Lambda}\right) \right)^2 + \left(\frac{\Lambda}{2k\pi} \cos\left(\frac{2k\pi T}{\Lambda}\right) - \frac{\Lambda}{2k\pi} \right)^2 \right) \\
 &= \frac{\Lambda}{2\pi^2 T^2} \sum_{k=1}^N \left(\frac{2 - 2 \cos\left(\frac{2k\pi T}{\Lambda}\right)}{k^2} \right) \leq \frac{2\Lambda}{\pi^2 T^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right), \quad (29)
 \end{aligned}$$

$$\begin{aligned}
R_1(t, s, \tau) &= \int_0^\Lambda H(v) r_N(t - s + \tau - v) dv \\
&= \frac{2}{\Lambda} \sum_{k=1}^N \left(\cos \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) \int_0^\Lambda H(v) \cos \left(\frac{2k\pi(s + v)}{\Lambda} \right) dv \right. \\
&\quad \left. + \sin \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) \int_0^\Lambda H(v) \sin \left(\frac{2k\pi(s + v)}{\Lambda} \right) dv \right) \\
&= \frac{2}{\Lambda} \sum_{k=1}^N \left(\cos \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) c_H(s) + \sin \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) s_H(s) \right) \\
&\leq \frac{I_1}{\pi} \sqrt{\frac{\Lambda}{2}} \sum_{k=1}^N \frac{\cos \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) + \sin \left(\frac{2k\pi(t + \tau)}{\Lambda} \right)}{k}. \quad (30)
\end{aligned}$$

Similarly,

$$R_2(t, s, \tau) \leq \frac{I_1}{\pi} \sqrt{\frac{\Lambda}{2}} \sum_{k=1}^N \frac{\cos \left(\frac{2k\pi(s + \tau)}{\Lambda} \right) + \sin \left(\frac{2k\pi(s + \tau)}{\Lambda} \right)}{k}. \quad (31)$$

By (23), (28), (29), (30), and (31) we have

$$\begin{aligned}
\text{Var} \hat{H}_{N,T,\Lambda}(\tau) &= \frac{1}{T^2} \left(\int_0^T \int_0^T R_H(t, s) r_N(t - s) dt ds \right. \\
&\quad \left. + \int_0^T \int_0^T R_1(t, s, \tau) \cdot R_2(t, s, \tau) dt ds \right) \\
&\leq \left(\frac{\Lambda^2 I_1^2}{2\pi^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right) \right) \cdot \left(\frac{2\Lambda}{\pi^2 T^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right) \right) \\
&\quad + \frac{1}{T^2} \left(\frac{I_1}{\pi} \sqrt{\frac{\Lambda}{2}} \sum_{k=1}^N \frac{1}{k} \int_0^T \left(\cos \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) + \sin \left(\frac{2k\pi(t + \tau)}{\Lambda} \right) \right) dt \right)^2 \\
&\leq \frac{\Lambda^3 I_1^2}{\pi^4 T^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right)^2 + \frac{1}{T^2} \left(\frac{2I_1\Lambda}{\pi^2} \sqrt{\frac{\Lambda}{2}} \sum_{k=1}^N \frac{1}{k^2} \right)^2 \\
&\leq \frac{3\Lambda^3 I_1^2}{\pi^4 T^2} \left(\frac{\pi^2}{6} - \frac{1}{N} \right)^2. \quad (32)
\end{aligned}$$

The theorem is proven. \square

Remark 5. From Theorem 1, it directly follows that the estimator $\hat{H}_{N,T,\Lambda}(\tau)$ is asymptotically unbiased. Indeed, as $N \rightarrow \infty$, the upper bound for the bias in (15) tends to zero.

Furthermore, as both $N \rightarrow \infty$ and $T \rightarrow \infty$, the variance in (16) tends to zero. Combined with the asymptotic unbiasedness, this proves that the estimator $\hat{H}_{N,T,\Lambda}(\tau)$ is consistent in the mean square sense.

3. CONCLUSION

In this paper, we examined a time-invariant continuous linear system characterized by a real-valued impulse response function (IRF) defined on the finite interval $[0, \Lambda]$. We employed a cross-correlogram approach to estimate the unknown IRF, modeling the input signal as a zero-mean Gaussian stochastic process expanded into a trimmed Fourier series using a trigonometric orthonormal basis.

We derived explicit analytical expressions and upper bounds for the mathematical expectation and variance of the proposed estimator. Based on these characteristics, we established the fundamental statistical properties of the estimator: we proved its asymptotic unbiasedness as the cutoff level $N \rightarrow \infty$, and demonstrated its consistency in the mean square sense as both $N \rightarrow \infty$ and the averaging interval $T \rightarrow \infty$.

For future research, we plan to explore scenarios involving discretely observed processes, which represent a more realistic setting in functional data analysis and practical applications.

The authors declare that there is no conflict of interest concerning the publication of this article.

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