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ПРОБЛЕМИ СТІЙКОСТІ ТА ЗБІЖНОСТІ ДИНАМІЧНИХ ПРОЦЕСІВ У НЕЙРОДИНАМІЦІ

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PROBLEMS OF STABILITY AND CONVERGENCE OF DYNAMIC PROCESSES IN NEURODYNAMICS

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ПРОБЛЕМИ СТІЙКОСТІ ТА ЗБІЖНОСТІ ДИНАМІЧНИХ ПРОЦЕСІВ У НЕЙРОДИНАМІЦІ

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ABSTRACT. The aim of the article is to derive sufficient conditions for stability and convergence of solutions to systems of differential equations that model neurodynamic processes, in particular the learning and operation dynamics of Hopfield neural networks. The analysis of asymptotic stability and convergence is based on Lyapunov's second method. Appropriate Lyapunov functions are constructed for the considered models, estimates for the derivative along trajectories are obtained, and conditions are formulated under which trajectories approach stationary regimes. The proposed approach extends the classical stability theory, which typically focuses on the stability of an individual trajectory (most often an equilibrium point), to the case of neurodynamic processes. Sufficient conditions for asymptotic stability of equilibrium states and for convergence of solutions to these equilibria are derived for systems of differential equations describing neurodynamic dynamics. For Hopfield network models, criteria are established that guarantee the decrease of a corresponding Lyapunov functional and, consequently, the convergence of learning and operating processes to stationary regimes. The obtained conditions are presented in a form suitable for verification in terms of model parameters and properties of the interconnection matrix. The derived stability and convergence criteria can be used in the design and tuning of Hopfield neural networks and related neurodynamic models to ensure guaranteed convergence of the learning process and stability of operating modes, as well as for parameter selection and validation in applied problems of information processing and optimization.

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KEYWORDS: differential equation systems, stability, neurodynamics, Hopfield networks.

АНОТАЦІЯ. Метою статті є отримання достатніх умов стійкості та збіжності розв'язків систем диференціальних рівнянь, що моделюють процеси нейродинаміки, зокрема процеси навчання та функціонування нейронних мереж Гопфілда. Для аналізу асимптотичної стійкості та збіжності використано апарат другого методу Ляпунова: побудовано функції Ляпунова для відповідних моделей, отримано оцінки похідної вздовж траєкторій та сформульовано умови, за яких траєкторії прямують до стаціонарних режимів. Узагальнено підхід класичної теорії стійкості, яка зазвичай зосереджується на стійкості окремої траєкторії (переважно точки рівноваги), на випадок динамічних процесів нейродинаміки. Виведено достатні умови асимптотичної стійкості рівноважних станів та умов збіжності розв'язків до них для систем диференціальних рівнянь, що описують нейродинамічні процеси. Для моделей нейронних мереж Гопфілда встановлено критерії, які гарантують спадання відповідного функціоналу Ляпунова та, як наслідок, збіжність процесів навчання і функціонування мережі до стаціонарних режимів. Отримані умови подано у формі, придатній для перевірки за параметрами моделі та характеристиками матриці зв'язків. Сформульовані критерії стійкості та збіжності можуть бути використані під час проектування і налаштування нейронних мереж Гопфілда та споріднених нейродинамічних моделей для забезпечення гарантованої збіжності процесу навчання й стабільності режимів роботи, а також для аналізу коректності вибору параметрів у прикладних задачах обробки інформації та оптимізації.

КЛЮЧОВІ СЛОВА: диференціальні рівняння, стійкість, нейродинаміка, мережа Гопфілда.

1. INTRODUCTION

The area of research in which neural networks are studied as nonlinear dynamical systems, with primary emphasis on stability and convergence problems, is referred to as neurodynamics. Classical stability theory typically addresses the stability of an individual trajectory, usually an equilibrium point. Accordingly, most traditional models focus on dynamical systems — differential or difference equations — with a single equilibrium.

When systems possess multiple equilibrium points, the concept of structurally stable (rough) systems arises. While necessary and sufficient conditions for structural stability are known for planar nonlinear systems, no such general criteria exist for high-dimensional systems. The systems of equations that describe neurodynamic processes are typically of very high dimension, which makes the direct application of classical stability methods problematic. In particular, traditional analytical techniques face fundamental difficulties caused by high dimensionality.

For this reason, neurodynamic models are often treated as weakly nonlinear systems, that is, systems with a distinguished linear part and nonlinearities confined to a sufficiently small sector.

In the case of sufficiently weak nonlinearity, the global stability of the equilibrium of the nonlinear system follows from the asymptotic stability of its linear approximation. Consequently, the convergence of learning processes can also be guaranteed.

The next important aspect in the study of neurodynamic processes is the presence of an aftereffect, or time delay. Real processes do not occur instantaneously; therefore, neglecting delay effects may lead to incorrect conclusions when analysing the dynamics of learning and pattern recognition [1].

In this paper, we derive conditions for the stability and convergence of solutions to systems of differential equations that describe neurodynamic processes.

For the study of the asymptotic stability of learning processes and the functioning of Hopfield neural networks, Lyapunov's second method is employed. The analysis is carried out in two stages: first, the Hopfield model is considered; second, the corresponding class of nonlinear dynamical systems in the space $\mathbb{R}^n$ is investigated.

2. LITERATURE REVIEW

The foundational neurodynamic model of associative memory is the Hopfield neural network, introduced and analysed from a dynamical-systems viewpoint in [1]. Approaches to global stability of weakly nonlinear systems under sector constraints and Lyapunov-function constructions of the Lurie–Postnikov type are widely used in nonlinear control and neurodynamics, and their computational verification aspects motivate optimization-based formulations; see, for instance, [2, 3].

2. THE AIM OF THE ARTICLE

The aim of the article is to derive conditions for the stability and convergence of solutions to systems of differential equations that describe neurodynamic processes, in particular the learning and operation dynamics of Hopfield neural networks.

3. METHODS

The study is based on Lyapunov's second method. The models are reduced to perturbation form near the equilibrium by a translation of variables, and nonlinearities are assumed to satisfy sector conditions.

A Lyapunov function of the Lurie–Postnikov type is constructed as a sum of quadratic terms and integral terms.

The total derivative of the Lyapunov function along trajectories is rewritten as a quadratic form involving an augmented matrix, and sufficient conditions for global asymptotic stability are obtained from the positive definiteness of a matrix sum that incorporates additional sector constraints.

4. RESULTS

1. Hopfield Model. The Hopfield neural network consists of a set of neurons forming a system with multiple feedback connections, where the number of feedbacks equals the number of neurons. The output of each neuron is fed, through a unit delay element, to all other neurons in the network, while self-feedback connections are absent [1, p. 856].

Under these assumptions, the dynamical equations of the network can be written in the form

$$C_i \dot{v}_i(t) = -\frac{v_i(t)}{R_i} + \sum_{j=1}^n \omega_{ij} \varphi_j(v_j(t)) + I_i, \quad (1)$$

A Hopfield network possesses a dynamical equilibrium if its input corresponds to a pattern that is mapped onto itself. From a physical viewpoint, the total energy of the system at such a state attains a minimum. The corresponding energy function, which serves as a Lyapunov function, can be represented as the sum of kinetic and potential energy terms,

$$V(y) = \frac{1}{2} \sum_{i,j=1}^n w_{ij} y_i y_j + \sum_{k=1}^n \delta_k y_k, \quad (2)$$

As follows from the theorems of Lyapunov's second method, the stationary state of the network is asymptotically stable if the Lyapunov function is positive definite and its total derivative along the system trajectories is negative definite.

In this paper, we investigate continuous Hopfield neural networks (CHNNs) without delay of the form [1]

$$C_i \dot{y}_i(t) = -\frac{1}{R_i} y_i(t) + \sum_{j=1}^n v_{ij} \varphi(y_j(t)) + I_i, \quad t \geq 0, \quad i = 1, \dots, n,$$

whereas, if the signal processing time is taken into account, systems of differential-difference equations with delay are considered,

$$C_i \dot{y}_i(t) = -\frac{1}{R_i} y_i(t) + \sum_{j=1}^n v_{ij} \varphi(y_j(t - \tau)) + I_i, \quad t \geq 0, \quad i = 1, \dots, n. \quad (3)$$

Here n denotes the number of neurons in the network, vector

$$y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$$

is the network state vector at time $t > 0$, and R_i , C_i , and I_i , $i = 1, \dots, n$, are the resistance, capacitance, and external input current of the i -th neuron, respectively.

By introducing the substitutions

$$a_i = \frac{1}{R_i C_i}, \quad b_{ij} = \frac{v_{ij}}{C_i}, \quad c_i = \frac{I_i}{C_i}, \quad i, j = 1, \dots, n,$$

systems (1) and (2) can be reduced to the standard form

$$\dot{y}_i(t) = -a_i y_i(t) + \sum_{j=1}^n b_{ij} \varphi(y_j(t)) + c_i, \quad t \geq 0, \quad i = 1, \dots, n.$$

If the delay time is taken into account, the corresponding system takes the form

$$\dot{y}_i(t) = -a_i y_i(t) + \sum_{j=1}^n b_{ij} \varphi(y_j(t - \tau)) + c_i, \quad t \geq 0, \quad i = 1, \dots, n.$$

2. Systems in $\mathbb{R}^n$. In the general case, a dynamical model of a neural network defined in the space $\mathbb{R}^n$ is considered. It is described by a system of nonlinear ordinary differential equations of the form

$$\begin{cases} \dot{y}_1 = -a_{11}y_1 + b_{11}f_{11}(y_1) + b_{12}f_{12}(y_2) + \dots + b_{1n}f_{1n}(y_n) + c_1, \\ \dot{y}_2 = -a_{22}y_2 + b_{21}f_{21}(y_1) + b_{22}f_{22}(y_2) + \dots + b_{2n}f_{2n}(y_n) + c_2, \\ \vdots \\ \dot{y}_n = -a_{nn}y_n + b_{n1}f_{n1}(y_1) + b_{n2}f_{n2}(y_2) + \dots + b_{nn}f_{nn}(y_n) + c_n. \end{cases} \quad (4)$$

We assume that $a_{ii} > 0$ for $i = 1, \dots, n$, the functions $f_{ij}(y_j)$, $i, j = 1, \dots, n$, are not necessarily differentiable, and that system (4) has a unique equilibrium point

$$M_0 = (y_1^0, y_2^0, \dots, y_n^0),$$

which is determined as the solution of the algebraic system

$$\begin{cases} -a_{11}y_1 + b_{11}f_{11}(y_1) + b_{12}f_{12}(y_2) + \dots + b_{1n}f_{1n}(y_n) + c_1 = 0, \\ -a_{22}y_2 + b_{21}f_{21}(y_1) + b_{22}f_{22}(y_2) + \dots + b_{2n}f_{2n}(y_n) + c_2 = 0, \\ \vdots \\ -a_{nn}y_n + b_{n1}f_{n1}(y_1) + b_{n2}f_{n2}(y_2) + \dots + b_{nn}f_{nn}(y_n) + c_n = 0. \end{cases}$$

To study the stability of the equilibrium point M_0 , we perform a translation of the coordinate system that shifts M_0 to the origin by introducing the change of variables

$$y_i = x_i + y_i^0, \quad i = 1, \dots, n. \quad (5)$$

After the change of variables (5), system (4) is transformed into a system of perturbation equations of the form

$$\begin{cases} \dot{x}_1 = -a_{11}x_1 + b_{11}F_{11}(x_1) + b_{12}F_{12}(x_2) + \dots + b_{1n}F_{1n}(x_n), \\ \dot{x}_2 = -a_{22}x_2 + b_{21}F_{21}(x_1) + b_{22}F_{22}(x_2) + \dots + b_{2n}F_{2n}(x_n), \\ \vdots \\ \dot{x}_n = -a_{nn}x_n + b_{n1}F_{n1}(x_1) + b_{n2}F_{n2}(x_2) + \dots + b_{nn}F_{nn}(x_n). \end{cases} \quad (6)$$

Here the nonlinear functions F_{ij} are defined by

$$F_{ij}(x_j) = f_{ij}(x_j + y_j^0) - f_{ij}(y_j^0), \quad i, j = 1, \dots, n.$$

By construction, $F_{ij}(0) = 0$ for all $i, j = 1, \dots, n$.

Thus, the problem of studying the stability of the equilibrium point

$$M_0 = (y_1^0, y_2^0, \dots, y_n^0)$$

and the convergence of solutions to this point is reduced to the analysis of the stability of the zero equilibrium point

$$O = (0, 0, \dots, 0)$$

of system (6).

Assume that the functions $F_{ij}(x_j)$, $i, j = 1, \dots, n$, satisfy the so-called *sector conditions*:

$$0 < F_{ij}(x_j) < L_{ij}x_j \quad \text{for } x_j > 0, \quad L_{ij}x_j < F_{ij}(x_j) < 0 \quad \text{for } x_j < 0, \quad F_{ij}(0) = 0.$$

These conditions can be written in a more compact form as

$$(L_{ij}x_j - F_{ij}(x_j))F_{ij}(x_j) > 0, \quad i, j = 1, \dots, n. \quad (7)$$

If conditions (7) are satisfied, sufficient conditions for asymptotic stability and convergence estimates can be obtained using a Lyapunov function of the Lurie–Postnikov type. Since the linear part of system (6) is diagonal, the Lyapunov function is constructed as a sum of quadratic terms and integral terms with constant coefficients,

$$V(x_1, x_2, \dots, x_n) = \sum_{i=1}^n h_{ii}x_i^2 + \sum_{i=1}^n \int_0^{x_i} \sum_{j=1}^n \gamma_{ji}F_{ji}(s) ds, \quad h_{ii} > 0. \quad (8)$$

Consider systems subject to the sector constraints (7). System (6) can be rewritten in the form

$$\dot{x}_i(t) = -a_{ii}x_i(t) + \sum_{j=1}^n b_{ij}F_j(x_j(t)), \quad i = 1, \dots, n. \quad (9)$$

For system (9), we choose the Lyapunov function in the form (8).

Let us introduce the following notation

$$F_2(t) = \begin{pmatrix} F_{21}(x_1(t)) \\ F_{22}(x_2(t)) \\ \vdots \\ F_{2n}(x_n(t)) \end{pmatrix}, \quad \dots, \quad F_n(t) = \begin{pmatrix} F_{n1}(x_1(t)) \\ F_{n2}(x_2(t)) \\ \vdots \\ F_{nn}(x_n(t)) \end{pmatrix},$$

$$S(h, \gamma) = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1(n+1)} \\ S_{12}^T & S_{22} & S_{23} & \cdots & S_{2(n+1)} \\ S_{13}^T & S_{23}^T & S_{33} & \cdots & S_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{1(n+1)}^T & S_{2(n+1)}^T & S_{3(n+1)}^T & \cdots & S_{(n+1)(n+1)} \end{pmatrix},$$

$$S_{11}(h) = \begin{pmatrix} 2h_{11}a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 2h_{22}a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 2h_{33}a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2h_{nn}a_{nn} \end{pmatrix},$$

$$\begin{aligned}
 S_{22}(\gamma) &= \begin{pmatrix} -\gamma_{11}b_{11} & -\frac{1}{2}\gamma_{11}b_{12} & 0 & \cdots & -\frac{1}{2}\gamma_{11}b_{1n} \\ -\frac{1}{2}\gamma_{11}b_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\gamma_{11}b_{1n} & 0 & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{33}(\gamma) &= \begin{pmatrix} 0 & -\frac{1}{2}\gamma_{22}b_{21} & 0 & \cdots & 0 \\ -\frac{1}{2}\gamma_{22}b_{21} & -\gamma_{22}b_{22} & 0 & \cdots & -\frac{1}{2}\gamma_{22}b_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\frac{1}{2}\gamma_{22}b_{2n} & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{(n+1)(n+1)}(\gamma) &= \begin{pmatrix} 0 & 0 & \cdots & -\frac{1}{2}\gamma_{nn}b_{n1} \\ 0 & 0 & \cdots & -\frac{1}{2}\gamma_{nn}b_{n2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\gamma_{nn}b_{n1} & -\frac{1}{2}\gamma_{nn}b_{n2} & \cdots & -\gamma_{nn}b_{nn} \end{pmatrix}, \\
 S_{12} &= \begin{pmatrix} \frac{1}{2}\gamma_{11}a_{11} - h_{11}b_{11} & -h_{11}b_{12} & \cdots & -h_{11}b_{1n} \\ 0 & \frac{1}{2}\gamma_{12}a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}\gamma_{1n}a_{nn} \end{pmatrix}, \\
 S_{13}(h, \gamma) &= \begin{pmatrix} \frac{1}{2}\gamma_{21}a_{11} & 0 & \cdots & 0 \\ -h_{22}b_{21} & \frac{1}{2}\gamma_{22}a_{22} - h_{22}b_{22} & \cdots & -h_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}\gamma_{2n}a_{nn} \end{pmatrix}, \\
 S_{1(n+1)}(h, \gamma) &= \begin{pmatrix} \frac{1}{2}\gamma_{n1}a_{11} & 0 & \cdots & 0 \\ 0 & \frac{1}{2}\gamma_{n2}a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -h_{nn}b_{n1} & -h_{nn}b_{n2} & \cdots & \frac{1}{2}\gamma_{nn}a_{nn} - h_{nn}b_{nn} \end{pmatrix}, \\
 S_{23}(\gamma) &= \begin{pmatrix} -\gamma_{21}b_{11} & 0 & \cdots & 0 \\ -\frac{1}{2}\gamma_{21}b_{12} - \frac{1}{2}\gamma_{12}b_{21} & -\frac{1}{2}\gamma_{12}b_{22} & \cdots & -\frac{1}{2}\gamma_{12}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\gamma_{21}b_{1n} & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{2(n+1)}(\gamma) &= \begin{pmatrix} -\gamma_{n1}b_{11} & 0 & \cdots & 0 \\ -\frac{1}{2}\gamma_{n1}b_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\gamma_{n1}b_{1n} - \frac{1}{2}\gamma_{1n}b_{n1} & -\frac{1}{2}\gamma_{1n}b_{n2} & \cdots & -\frac{1}{2}\gamma_{1n}b_{nn} \end{pmatrix}, \\
 S_{3(n+1)}(\gamma) &= \begin{pmatrix} 0 & -\frac{1}{2}\gamma_{n2}b_{21} & \cdots & 0 \\ 0 & -\frac{1}{2}\gamma_{n1}b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\gamma_{2n}b_{n1} & -\frac{1}{2}\gamma_{2n}b_{n2} - \frac{1}{2}\gamma_{2n}b_{n2} & \cdots & -\frac{1}{2}\gamma_{2n}b_{nn} \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 S_1 &= \begin{pmatrix} S_{11}^1 & S_{12}^1 & S_{13}^1 & \cdots & S_{1(n+1)}^1 \\ (S_{12}^1)^T & S_{22}^1 & S_{23}^1 & \cdots & S_{2(n+1)}^1 \\ (S_{13}^1)^T & (S_{23}^1)^T & S_{33}^1 & \cdots & S_{3(n+1)}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (S_{1(n+1)}^1)^T & (S_{2(n+1)}^1)^T & (S_{3(n+1)}^1)^T & \cdots & S_{(n+1)(n+1)}^1 \end{pmatrix}, \\
 S_{11}^1 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad S_{22}^1 = \begin{pmatrix} \omega_{11} & 0 & \cdots & 0 \\ 0 & \omega_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{1n} \end{pmatrix}, \\
 S_{33}^1 &= \begin{pmatrix} \omega_{21} & 0 & \cdots & 0 \\ 0 & \omega_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{2n} \end{pmatrix}, \quad S_{(n+1)(n+1)}^1 = \begin{pmatrix} \omega_{n1} & 0 & \cdots & 0 \\ 0 & \omega_{n2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{nn} \end{pmatrix}, \\
 S_{23}^1 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad S_{2(n+1)}^1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad S_{3(n+1)}^1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{12}^1 &= \begin{pmatrix} -\frac{1}{2}\omega_{11}L_{11} & -\frac{1}{2}\omega_{12}L_{12} & \cdots & -\frac{1}{2}\omega_{1n}L_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{13}^1 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\frac{1}{2}\omega_{21}L_{21} & -\frac{1}{2}\omega_{22}L_{22} & \cdots & -\frac{1}{2}\omega_{2n}L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\
 S_{1(n+1)}^1 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\omega_{n1}L_{n1} & -\frac{1}{2}\omega_{n2}L_{n2} & \cdots & -\frac{1}{2}\omega_{nn}L_{nn} \end{pmatrix}. \tag{10}
 \end{aligned}$$

The following result holds.

Theorem 1. *Let there exist constants h_i , γ_{ij} , and ω_{ij} , $i, j = 1, \dots, n$, such that the matrix $(S + S_1)$ is positive definite. Then the zero solution of system (9) is globally asymptotically stable.*

Proof. It follows from representation (8) that the function $V(x_1, x_2, \dots, x_n)$ is positive definite. Let us compute its total derivative along the trajectories of the system.

It has the form

$$\begin{aligned} \frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) &= \frac{\partial V(x_1(t), x_2(t), \dots, x_n(t))}{\partial x_1(t)} \dot{x}_1(t) \\ &+ \frac{\partial V(x_1(t), x_2(t), \dots, x_n(t))}{\partial x_2(t)} \dot{x}_2(t) + \dots + \frac{\partial V(x_1(t), x_2(t), \dots, x_n(t))}{\partial x_n(t)} \dot{x}_n(t). \end{aligned}$$

Or

$$\begin{aligned} \frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) &= \\ &= \left(2h_{11}x_1(t) + \gamma_{11}F_{11}(x_1(t)) + \gamma_{21}F_{21}(x_1(t)) + \dots + \gamma_{n1}F_{n1}(x_1(t)) \right) \\ &\times \left(-a_{11}x_1(t) + b_{11}F_{11}(x_1(t)) + b_{12}F_{12}(x_2(t)) + \dots + b_{1n}F_{1n}(x_n(t)) \right) \\ &+ \left(2h_{22}x_2(t) + \gamma_{12}F_{12}(x_2(t)) + \gamma_{22}F_{22}(x_2(t)) + \dots + \gamma_{n2}F_{n2}(x_2(t)) \right) \\ &\times \left(-a_{22}x_2(t) + b_{21}F_{21}(x_1(t)) + b_{22}F_{22}(x_2(t)) + \dots + b_{2n}F_{2n}(x_n(t)) \right) \\ &\quad + \dots + \\ &+ \left(2h_{nn}x_n(t) + \gamma_{1n}F_{1n}(x_n(t)) + \gamma_{2n}F_{2n}(x_n(t)) + \dots + \gamma_{nn}F_{nn}(x_n(t)) \right) \\ &\times \left(-a_{nn}x_n(t) + b_{n1}F_{n1}(x_1(t)) + b_{n2}F_{n2}(x_2(t)) + \dots + b_{nn}F_{nn}(x_n(t)) \right). \end{aligned}$$

By expanding the corresponding expressions, we obtain the following equality:

$$\frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) = \Sigma_1 + \Sigma_2 + \dots + \Sigma_n,$$

where

$$\begin{aligned} \Sigma_1 &= -2h_{11}a_{11}x_1^2(t) - a_{11}x_1(t) \sum_{j=1}^n \gamma_{j1}F_{j1}(x_1(t)) \\ &\quad + 2h_{11}b_{11}x_1(t)F_{11}(x_1(t)) + b_{11} \sum_{j=1}^n \gamma_{j1}(F_{j1}(x_1(t)))^2 \\ &\quad + \sum_{k=2}^n \left(2h_{kk}b_{kk}x_k(t)F_{kk}(x_k(t)) + b_{kk} \sum_{j=1}^n \gamma_{jk}(F_{jk}(x_k(t)))^2 \right) \\ &\quad + \sum_{k=2}^n \sum_{j=1}^n \gamma_{j1}b_{1k}F_{jk}(x_k(t)) F_{j1}(x_1(t)), \end{aligned}$$

$$\begin{aligned}\Sigma_2 = & -2h_{22}a_{22}x_2^2(t) - a_{22}x_2(t) \sum_{j=1}^n \gamma_{j2}F_{j2}(x_2(t)) \\ & + \sum_{k=1}^n \left(2h_{22}b_{2k}x_2(t) F_{2k}(x_k(t)) + b_{2k} \sum_{j=1}^n \gamma_{j2}F_{2k}(x_k(t)) F_{j2}(x_2(t)) \right),\end{aligned}$$

and

$$\begin{aligned}\Sigma_n = & -2h_{nn}a_{nn}x_n^2(t) - a_{nn}x_n(t) \sum_{j=1}^n \gamma_{jn}F_{jn}(x_n(t)) \\ & + \sum_{k=1}^n \left(2h_{nn}b_{nk}x_n(t) F_{nk}(x_k(t)) + b_{nk} \sum_{j=1}^n \gamma_{jn}F_{nk}(x_k(t)) F_{jn}(x_n(t)) \right).\end{aligned}$$

Using the notation (10), we rewrite the expression for the total derivative of the Lyapunov function in the quadratic form

$$\begin{aligned}& \frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) \\ = & -(x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t)) S (x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t))^T.\end{aligned}$$

However, due to the introduction of the variables $x(t)$ and $F_i(t)$, $i = 1, \dots, n$, the dimension of the matrix S increases significantly, and even in the scalar case this matrix is not positive definite. Therefore, we employ the additional sector conditions (7) and rewrite the total derivative of the Lyapunov function for $i, j = 1, \dots, n$, $\omega_{ij} > 0$, in the form

$$\begin{aligned}\frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) = & - \begin{pmatrix} x(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}^T S \begin{pmatrix} x(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} \\ & + \sum_{i,j=1}^n \omega_{ij} (L_{ij}x_j - F_{ij}(x_j)) F_{ij}(x_j) - \sum_{i,j=1}^n \omega_{ij} (L_{ij}x_j - F_{ij}(x_j)) F_{ij}(x_j).\end{aligned}$$

Introducing the first sum into the quadratic form, we obtain

$$\begin{aligned}\frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) \\ = & -(x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t)) \\ & \times (S + S_1) (x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t))^T \\ & - \sum_{i,j=1}^n \omega_{ij} (L_{ij}x_j - F_{ij}(x_j)) F_{ij}(x_j), \quad \omega_{ij} > 0. \quad (11)\end{aligned}$$

Using the sector conditions (7), the second term in (11) is nonpositive. Hence, we obtain the inequality

$$\begin{aligned} \frac{d}{dt}V(x_1(t), x_2(t), \dots, x_n(t)) \\ \leq -(x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t)) \\ \times (S + S_1) (x^T(t), F_1^T(t), F_2^T(t), \dots, F_n^T(t))^T. \end{aligned}$$

If the matrix $S + S_1$ is positive definite, then the total derivative of the Lyapunov function is negative definite. Consequently, the zero solution of the system of perturbation equations is asymptotically stable. $\square$

Remark 1. It should be noted that the total derivative of the Lyapunov function depends both on the system parameters, which are specified in advance, and on the parameters h_i , γ_{ij} , and ω_{ij} , $i, j = 1, \dots, n$, which can be chosen arbitrarily in the process of establishing stability. Therefore, finding explicit stability conditions for the zero equilibrium point of the system, which corresponds to the learning condition, reduces to determining conditions under which the matrix sum $S + S_1$ is maximally positive definite. Hence, the problem of stability analysis, that is, the problem of convergence of learning processes, can be reduced to a quadratic optimization problem [2, 3].

5. DISCUSSION

- The main finding is a Lyapunov-based sufficient condition ensuring global asymptotic stability of the zero solution of the perturbation system under sector constraints, formulated via positive definiteness of the matrix $(S + S_1)$.
- The results are consistent with the classical Hopfield-network viewpoint [1] and complement related optimization-oriented approaches to stability verification [2, 3].
- A limitation is that the conditions are sufficient and may be conservative; additionally, the verification and maximization of positive definiteness of $(S + S_1)$ may be computationally demanding for very large number n .
- These limitations motivate future research on less conservative criteria, scalable numerical tests, and an explicit extension of the presented matrix conditions to delayed neurodynamic models.
- The obtained conditions provide a constructive guideline for selecting parameters and designing interconnection structures that guarantee convergence of learning dynamics.

6. CONCLUSION

Sufficient Lyapunov-type conditions for stability and convergence of neurodynamic models are derived; in particular, global asymptotic stability is guaranteed when the matrix $(S + S_1)$ is positive definite under the sector constraints.

USE OF AI TOOLS

During manuscript preparation, the author used ChatGPT (OpenAI, model GPT-5.2 Thinking) for language editing, grammar checking, and text structuring (formatting). The author reviewed all changes and takes full responsibility for the content and correctness of the manuscript.

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